Partial Differential Equation

The mathematical formulations of many problems in science and engineering reduce to study of first-order PDEs. For instance, the study of first-order PDEs arise in gas flow problems, traffic flow problems, phenomenon of shock waves, the motion of wave fronts, It is therefore essential to study the theory of first-order PDEs and the nature their solutions to analyze the related physical problems. we shall study first-order linear, and nonlinear PDEs and methods of solving these equations. An important method of characteristics is explained for these equations in which solving PDE reduces to solving an ODE system along a characteristics curve. Further, the Charpit's method for nonlinear first-order PDEs are discussed.

INTRODUCTION

A differential equation involving partial derivatives of a dependent variable(one or more) with more than one independent variable is called a partial differential equation, hereafter denoted as PDE.

Consider the following equations:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \textbf{(1.1)}$$

$$\frac{\partial u}{\partial t} - T \frac{\partial^2 u}{\partial x^2} = 0 \qquad \textbf{(1.2)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \textbf{(1.3)}$$

$$\left(x^2 + y^2\right)\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x \partial y} - 3u = 0 \qquad \textbf{(1.4)}$$

$$ux \frac{\partial^2 u}{\partial x^2} + u^2 xy \frac{\partial^2 u}{\partial x \partial y} + uy \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u^3 = 0 \qquad \textbf{(1.5)}$$

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2 \qquad \textbf{(1.6)}$$

Order of a PDE: The order of the highest derivative term in the equation is called the order of the PDE. Thus equations (1.1 to 1.6) are all of second order.

Linear PDE: If the dependent variable and all its partial derivatives occure linearly in any PDE then such an equation is called linear PDE otherwise a non-linear PDE. In the above example equations 1.1, 1.2, 1.3 & 1.4 are linear whereas 1.5 & 1.6 are non-linear.

Lecture 1 First-Order Partial Differential Equations

A first order PDE in two independent variables x, y and the dependent variable z can be written in the form

$$f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0 \qquad (1)$$

For Convenience we set $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$

Then equation (1) becomes

$$f(x, y, z, p, q) = 0$$

Formation of first-order PDEs

Method I (Eliminating arbitrary constants) : Consider two parameters family of sur-faces described by the equation F(x,y,z,a,b) = 0, (2) where a and b are arbitrary constants.

Differentiating equation (2) with respect to x and y, we obtain

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0$$

By eliminating constants a & b we will obtain equation of the form

$$f(x, y, z, p, q) = 0$$

Examples:

Q.1 Form the Partial Differential equation of: $z = ax + by + ab$

Ans. $p = \frac{\partial z}{\partial x} = a$; $q = \frac{\partial z}{\partial y} = b$

Hence $z = px + qy + pq$.

Q.2 Form the partial differential equation of $z = (x+a)(y+b)$.

Ans. $\frac{\partial z}{\partial x} = (y+b)$ $\Rightarrow p = y + b$

$$\frac{\partial z}{\partial y} = (y+a) \qquad \Rightarrow q = x+a$$

Hence $z = pq$

Q.3 Form the partial differential equation of $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Ans. Partial differentiating w.r.t. x,

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \qquad \Rightarrow \frac{1}{a^2} = \frac{p}{x}$$

$$2\frac{\partial z}{\partial y} = \frac{2y}{b^2} \qquad \Rightarrow \frac{1}{b^2} = \frac{q}{y}$$

Hence $2z = px + qy$

Q.4 Form the partial differential equation of $(x-h)^2 + (y-k)^2 + z^2 = a^2$.

Ans. Partial differentiating w.r.t. x,

$$2(x-h) + 2z\frac{\partial z}{\partial x} = 0 \qquad \Rightarrow x-h = -zp$$

Partial differentiating w.r.t. y,

$$2(y-k) + 2z\frac{\partial z}{\partial y} = 0 \qquad \Rightarrow y-k = -zq$$

Hence $z^2(p^2 + q^2 + 1) = a^2$

Method II (Eliminating arbitrary Functions):

Q.5 Form the partial differential equation of $z = f(x^2 + y^2)$.(2 marks)

Ans. Partial differentiating w.r.t. x,

$$\frac{\partial z}{\partial x} = f'\left(x^2 + y^2\right) \cdot 2x$$

Partial differentiating w.r.t. y,

$$\frac{\partial z}{\partial y} = f'\left(x^2 + y^2\right) \cdot 2y$$

Hence $\frac{p}{q} = \frac{x}{y}$

Q. 6 Form a partial differential equation of $z = xf_1(x+t) + f_2(x+t)$ (4 marks)

Ans. Partial differentiate w.r.t x

$$\frac{\partial z}{\partial x} = f_1(x+t) + xf_1'(x+t) + f_2'(x+t)$$

$$\frac{\partial^2 z}{\partial x^2} = f_1^1(x+t) + f_1^1(x+t) + xf_1''(x+t) + f_2''(x+t)$$

$$\frac{\partial^2 z}{\partial x^2} = 2f_1^1(x+t) + xf_1''(x+t) + f_2''(x+t) \quad (1)$$

$$\frac{\partial z}{\partial t} = xf_1'(x+t) + f_2'(x+t)$$

$\frac{\partial^2 z}{\partial t^2} = xf_1''(x+t) + f_2''(x+t)$ (2)

Subtracting (2) from (1)

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 2f_1^1(x+t)$$

Also $\frac{\partial^2 z}{\partial x \partial t} = f_1^1(x+t) + xf_1''(x+t) + f_2''(x+t) = \frac{1}{2}\left[\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2}\right] + \frac{\partial^2 z}{\partial t^2}$

Q.7 Form the partial differential equation of $f\left(x^2 + y^2, z - xy\right) = 0$. (4 marks)

Ans. Let

$$u = x^2 + y^2 \qquad v = z - xy$$

$$\boxed{\frac{\partial u}{\partial x} = 2x} \qquad \frac{\partial v}{\partial x} = \frac{\partial z}{\partial x} - y$$

$$\boxed{\frac{\partial u}{\partial y} = 2y} \qquad \boxed{\frac{\partial v}{\partial x} = P - y}$$

$$\frac{\partial v}{\partial y} = \frac{\partial z}{\partial y} - x$$

$$\boxed{\frac{\partial v}{\partial y} = q - x}$$

Hence $f(u,v) = 0$

Partial Diff. w.r.t. *x*, $\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = 0$

$$\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(P - y) = 0 \text{ ....(1)}$$

Partial Diff. w.r.t. *y*, $\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = 0$

$$\frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}(q - x) = 0 \text{ ....(2)}$$

The two equations have the solution:

$$\begin{vmatrix} 2x & P - y \\ 2y & q - x \end{vmatrix} = 0$$

$$2x(q - x) - 2y(P - y) = 0$$

Method to Solve First order Linear Partial differential Equation

Explain the method to solve the Lagrange's Linear Equation.

Ans. The equation $Pp + Qq = R$

Auxiliary equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

Solve these equations by method of Grouping or method of multipliers and get $u=a, v=b$.

Hence the general solution of the equations is $\phi(u,v)=0$.

Q 1 Solve the equation $yzP+zxq=xy$.

Ans. $$\frac{dx}{yz}=\frac{dy}{zx}=\frac{dz}{xy}$$

$$\frac{dx}{y\not{z}}=\frac{dy}{\not{z}x}$$

$$\int xdx=\int ydy$$

$$x^2-y^2=c_1$$

$$\frac{dy}{z\not{x}}=\frac{dz}{\not{x}y}$$

$$\int ydy=\int zdz$$

$$y^2-z^2=c_2$$

Hence solution is $\phi\left(x^2-y^2, y^2-z^2\right)=0$

Q.2 Solve the partial differential equation $2p+3q=1$.

Ans. $$\frac{dx}{2}=\frac{dy}{3}=\frac{dz}{1}$$

$$\int\frac{dx}{2}=\int\frac{dy}{3}$$

$$3x-2y=c_1$$

$$\int\frac{dy}{3}=\int\frac{dz}{1}$$

$$y-3z=c_2$$

Hence solution is $\phi(3x-2y, y-3z)=0$.

Q.3 Solve the partial differential equation $xp+yq=3z$.

Ans. $$\frac{dx}{p}=\frac{dy}{y}=\frac{dz}{3z}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y} \qquad \Bigg| \qquad \int \frac{dy}{y} = \int \frac{dz}{3z}$$

$$\frac{x}{y} = c_1 \qquad \Bigg| \qquad \frac{y^3}{z} = c_2$$

Hence solution is $\phi\left(\frac{x}{y}, \frac{y^3}{3z}\right) = 0$.

Q.4 Solve $y^2zp + x^2zq = xy^2$.

Ans. $\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$

Its solution is $\phi\left(x^3 - y^3, x^2 - z^2\right) = 0$.

Homogeneous Linear Partial Differential Equations with constant coefficients

An equation of the form

$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots \ldots . + a_n \frac{\partial^n z}{\partial y^n} = F(x, y)$, where all $a_i's$ are constants is call homogeneous partial differential equation of nth order.

Method to Solve Homogeneous Linear Partial Differential Equations with constant coefficients:

Complete Solution = Complementary Function + Particular Integral

Rules to find C.F.:

Putting $D = \frac{\partial}{\partial x} = m$ and $D' = \frac{\partial}{\partial y} = 1$, we get the auxiliary equation (A.E) $\emptyset(m, 1) = 0$

Solve it for m

(i) If roots of the auxiliary equation are $m_1 m_2, \ldots \ldots \ldots$(all unequal, real or imaginary)
Then $C.F. = f_1(y + m_1x) + f_2(y + m_2x) + \cdots \ldots \ldots$

(ii) If two roots are equal Then $C.F. = f_1(y + \text{m}x) + xf_2(y + \text{m}x) + \cdots \ldots \ldots$

(iii) If three roots are equal the $C.F. = f_1(y + mx) + xf_2(y + mx) + x^2 f_3(y + mx)$ … … ….

Q.1 Solve the partial differential equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$.

Ans. Then the equation is $(D^2 - DD' - 6D'^2)z = 0$

A.E. is $m^2 - m - 6 = 0$ $\qquad D = m, D' = 1$

$m = 3, -2$

C.F. $f_1(y + 3x) + f_2(y - 2x)$

P.I. = 0

Hence the complete solution is $z = f_1(y + 3x) + f_2(y - 2x)$.

Q.2 Solve the partial differential equation $4r - 12s + 9t = 0$.

Ans. $(4D^2 - 12DD' + 9D'^2)z = 0$

A.E. is $4m^2 - 12m + 9 = 0$

$m = \frac{3}{2}, \frac{3}{2}$

C.F. $f_1\left(y + \frac{3x}{2}\right) + xf_2\left(y + \frac{3x}{2}\right)$

P.I. = 0

$$z = f_1\left(y + \frac{3x}{2}\right) + xf_2\left(y + \frac{3x}{2}\right).$$

Q.3 Solve the partial differential equation $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 2\frac{\partial^3 z}{\partial x \partial y^2} = 0$

Ans. $\left(D^3 - 3D^2 D' + 2DD'^2\right)z = 0$

A.E. is $m^3 - 3m^2 + 2m = 0$

$m = 0,1,2$

C.F. $f_1(y + 0x) + f_2(y + x) + f_3(y + 2x)$

P.I. = 0

$z = f_1(y) + f_2(y + x) + f_3(y + 2x).$

Rules to find P.I:

If $F(x, y) = e^{ax+by}$,then $P.I. = \frac{1}{\emptyset(D,D')} e^{ax+by} = \frac{1}{\emptyset(a,b)} e^{ax+by}$

Q.1 Solve the partial differential equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = e^{3x+2y}$.

Ans. $\left(D^2 + 2DD' + D'^2\right)z = e^{3x+2y}$

A.E. $m^2 + 2m + 1 = 0$

$m = -1,-1$

C.F. $f_1(y - x) + xf_2(y - x)$

P.I. $= \frac{1}{\left(D^2 + 2DD' + D'^2\right)} e^{3x+2y}$

$D = 3, D' = 2$

$3x + 2y = u$

$$= \frac{1}{(3^2+2\cdot3\cdot2+2^2)}\iint e^u du du$$

$$= \frac{e^{3x+2y}}{25}$$

If $F(x,y) = Sin(ax+by)$ or $Cos(ax+by)$ then $P.I = \frac{1}{\emptyset(D,D')} Sin(ax+by)$

Replace $D^2 by - a^2, DD' \; by - ab, D'^2 by - b^2$

Q.1 Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x\, \partial y} = \text{sinxcos2y}$

Sol. In symbolic form: $D^2 - DD')z = \frac{1}{2}.2sinx\ cos2y$

$$= \frac{1}{2}[\sin(x+2y) + \sin(x-2y)]$$

The A.E. is $m^2 - m = 0 \;\; or\; m(m-1) = 0 \;\; i.e. m = 0,1$

$$C.F. = f_1(y) + f_2(y+x)$$

$$P.I = \frac{1}{2}\frac{1}{D^2 - DD'}\sin(x+2y) + \frac{1}{2}\frac{1}{D^2 - DD'}\sin(x-2y)$$

Putting $D^2 = -a^2 = -1, DD' = -ab = -(1)(2) = -2\ in\ first\ term$ & $D^2 = -a^2 = -1, DD' = -ab = -(1)(-2) = 2\ in\ second\ term\ we\ get$

$$P.I = \frac{1}{2}\frac{1}{-1+2}.\sin(x+2y) + \frac{1}{2}\frac{1}{-1-2}\sin(x-2y)$$

$$= \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$$

Hence the complete sol. Is

$$z = f_1(y) + f_2(y+x) + \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$$

If $F(x,y) = x^m y^n$ then $P.I. = \frac{1}{\emptyset(D,D')} x^m y^n = [\emptyset(D,D')]^{-1} x^m y^n$

$$If\ m < n, expand\ \left[\emptyset(\boldsymbol{D}, \boldsymbol{D}')\right]^{-1} \ln powers\ of\ \frac{D}{D'}$$

$$If\ m > n, expand\ \left[\emptyset(\boldsymbol{D}, \boldsymbol{D}')\right]^{-1} \ln powers\ of\ \frac{D'}{D}$$

Q.15 Solve the Linear Partial Differential Equation $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$

Ans. $(D^2 + 3DD' + 2D'^2)z = x + y$

A.E. is $m^2 + 3m + 2 = 0$

$m = -1, -2$

C.F. $f_1(y - x) + f_2(y - 2x)$

P.I. $= \frac{1}{(D^2 + 3DD' + 2D'^2)}(x + y)$

$D = 1, D' = 1$

$x + y = u$

$$\frac{1}{(1 + 3 \cdot 1 \cdot 1 + 2 \cdot 1^2)} \iint u du du = \frac{(x + y)^3}{36}$$

Hence $z = f_1(y - x) + f_2(y - 2x) + \frac{(x + y)^3}{36}$.

Non Linear Equations of first order:

A partial Differential equation which involves first order partial derivatives p & q with degree higher than one and products of p & q is called non linear partial differential equation. The complete solution of such an equation involves only two arbitrary constants.

Equations of the form $\boldsymbol{f(p,q)=0}$

Eg. $pq = p + q$

The equation is of the form $f(p,q) = 0$

The C.F. is $z = ax + by + c$

Where $ab = a + b \; or \; b = \frac{a}{a-1}$

So complete solution is $z = ax + \frac{a}{a-1}y + c$

Equations of the form $\boldsymbol{f(z,p,q)=0}$

e.g. Solve $z^2(p^2 + q^2 + 1) = a^2$

The equation is of the form $f(z,p,q) = 0$

$let\; u = x + by$

So that $p = \frac{dz}{du}$ & $q = b\frac{dz}{du}$

Substituting these values of p &q in the given equation we get,

$$z^2\left[\left(\frac{dz}{du}\right)^2 + b^2\left(\frac{dz}{du}\right)^2 + 1\right] = a^2$$

Or $z^2(1 + b^2)\left(\frac{dz}{du}\right)^2 = a^2 - z^2$

Or $z\sqrt{1+b^2}\frac{dz}{du} = \pm\sqrt{a^2 - z^2}$

Or $\pm\sqrt{1+b^2}.\frac{z}{\sqrt{a^2-z^2}}dz = du$

Integrating we have

$\pm\sqrt{1+b^2}\sqrt{a^2 - z^2} = u + c$

Or $(1 + b^2)(a^2 - z^2) = (x + by + c)^2$

Which is the required solution

Equations of the form $\boldsymbol{f_1(x,p) = f_2(y,q)}$

e.g. $Solve\ yp = 2yx + logq$

$p = 2x + \frac{1}{y}logq$ or $p - 2x = \frac{1}{y}logq$

Which is of the form $f_1(x,p) = f_2(y,q)$

$$let\ p - 2x = \frac{1}{y}\log q = a,\ then\ p = 2x + a\ and\ logq = ay\ i.e.\ q = e^{ay}$$

Substituting the values of p & q in $dz = pdx + qdy, we\ get$

$dz = (2x + a)dx + e^{ay}\ dy$

Integrating we get,

$$z = x^2 + ax + \frac{1}{a}e^{ay}dy$$

Equations of the form $\boldsymbol{z = px + qy + f(p,q)[Clairout\ Equation]}$

C.F.is obtained by replacing a by p &b by q

e.g. Solve $(p - q)(z - px - qy) = 1$

$$z - px - qy = \frac{1}{p - q}$$

$$z = px + qy + \frac{1}{p - q}$$

Equations of the form $z = px + qy + f(p,q)[Clairout\ Equation]$

C. F. is $z = ax + by + \frac{1}{a-b}$

Charpit's Method: Let the given equation be $f(x,y,z,p,q) = 0$

The Auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial z}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dF}{0}$$

Any integral which involves p or q or both can be taken as the assumed relation and find the solutions

e.g. Solve $2zx-px^2-2qxy+pq=0$

Ans. $f=2zx-px^2-2qxy+pq=0$

$$\frac{\partial f}{\partial x}=2z-2px-2qy,\quad \frac{\partial f}{\partial y}=-2qx,\quad \frac{\partial f}{\partial z}=2x,\quad \frac{\partial f}{\partial p}=-x^2+q,\quad \frac{\partial f}{\partial q}=-2xy+p$$

Charpit's Auxiliary Equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial z}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dF}{0}$$

Or

$$\frac{dp}{2z-2qy}=\frac{dq}{0}=\frac{dz}{px^2-2pq+2qxy}=\frac{dx}{x^2-q}=\frac{dy}{2xy-p}=\frac{dF}{0}$$

Putting $q=a\ in\ the\ staement\ we\ get$

$$p=\frac{2x(z-ay)}{x^2-a}$$

$$dz=pdx+qdy=\frac{2x(z-ay)}{x^2-a}dx+ady$$

Or $\frac{dz-ady}{z-ay}=\frac{2x}{x^2-a}dx$

Integrating we get , $\log(z-ay)=\log(x^2-a)+logb$

Or $z-ay=b(x^2-a)$

Or $z=ay+b(x^2-a)$ which is the required solution.

Question Bank

Q.1 Form the partial differential equations by eliminating the arbitrary constants from the following :

(i) $Z = ax + by + a^2 + b^2$ (ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(iii) $z = a(x + y) + b(x - y) + abt + c$ (iv) $z = Ae^{pt} sinpx$

Q.2 Solve using Charpit Method :

(1) $z = pq$ (2) $z = px + qy + p^2 + q^2$

(3) $(p^2 + q^2)y = qz$ (4) $z^2(p^2z^2 + q^2)$